臺灣綜合大學系統105學年度學士班轉學生聯合招生考試試題

科目名稱	基礎數學	類組代碼	D25
		科目碼	D2591
※本項考試依簡章規定各考科均「不可以」使用計算機		本試題共計	2 頁

(請提供詳細計算或證明過程, 僅有答案而沒有過程得零分!)

1. (10分) Let $g(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

   Show that partial derivatives $g_x(0,0)$ and $g_y(0,0)$ exists while $g(x,y)$ is not continuous at (0,0).

2. (10分) Use the ε-δ definition to show that $f(x) = \sqrt{x+2}$ is continuous at $x = 2$.

3. (10分) Evaluate the integral $\int\int_D (x+y)e^{x^2-y^2}\, dA$, where D is the region enclosed by the lines $x - y = 0$, $x - y = 2$, $x + y = 0$ and $x + y = 3$.

4. (10分) Consider the power series $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$ and define the sequence of its partial sums

$$S_n(x) = \sum_{k=1}^{n} \frac{x^k}{k^2}.$$

   Find the interval $\mathcal{I}$ of absolute convergence for the power series and show that $S_n(x)$ converges uniformly to $f(x)$ on $\mathcal{I}$.

5. (6+6=12分) (a) Does the mean value theorem apply to $f(x) = \sqrt{|x|}$ on [-2,2]? Why?

   (b) Prove that if $g(x) \in C^3[a,b]$ and $g(a) = g'(a) = g''(a) = 0$ and $g(b) = 0$, then there is a number $c \in (a,b)$, with $g^{(3)}(c) = 0$.

背面有題，請繼續作答。

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6. (6+6=12分) (a) Find the limit: $\lim_{x\to 0} \dfrac{2x^2 \sin \frac{1}{x}}{\sin x}$.

   (b) Consider the nested intervals $[a_n, b_n]$:

$$a_n \leq a_{n+1} \leq \cdots \quad \leq \cdots b_{n+1} \leq b_n, \quad \text{for } n = 1, 2, 3, \cdots$$

   and $\lim_{n\to\infty} |a_n - b_n| = 0$. Prove that the sequence $\{a_n\}$ and $\{b_n\}$ converge to a unique real number α.

7. (6+6=12分) A sequence $\{a_n\} \subset \mathbb{R}$ is called a **Cauchy sequence** if for each $\varepsilon > 0$ there is a positive integer N such that $n, m \geq N$ implies $|a_n - a_m| < \varepsilon$.

   (a) Let $s_n = \sum_{k=1}^{n} \dfrac{1}{2k-1}$. Is $\{s_n\}$ a Cauchy sequence? Why?

   (b) Is every Cauchy sequence monotone? Why?

8. (6+6=12分) (a) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous on $\mathbb{R}$. Prove that $(f + g)$ is uniformly continuous on $\mathbb{R}$.

   (b) Is $f(x) = 1/x$ uniformly continuous on $(0, \infty)$? Why?

9. (6+6=12分) Consider sequence of functions $f_n(x) = \dfrac{nx}{nx+1}$, $x \in [0, \infty]$.

   (a) Find the pointwise limit of $f_n(x)$. Call this limit $f(x)$.

   (b) Does $f_n(x)$ converge to $f(x)$ uniformly on [0,1]? Why?