臺灣綜合大學系統 111 學年度學士班轉學生聯合招生考試試題

科目名稱	線性代數	類組代碼	A07.C11
		科目碼	A0702

※本項考試依簡章規定所有考科均「不可」使用計算機。　　本科試題共計 1 頁

Note: $\mathbb{R}$ denotes the field of real numbers.

1. Let $\mathrm{M}_{2\times 2}(\mathbb{R})$ be the real vector space consisting of all 2×2 real matrices, and let

$$W_1 = \left\{ \begin{bmatrix} a & -a \\ b & c \end{bmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{R}) \;:\; a, b, c \in \mathbb{R} \right\}.$$

(a) (8 points) Prove that W_1 is a subspace of $\mathrm{M}_{2\times 2}(\mathbb{R})$.

(b) (12 points) Let $W_2 = \left\{ \begin{bmatrix} a & b \\ -a & c \end{bmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{R}) \;:\; a, b, c \in \mathbb{R} \right\}$, which is also a subspace of $\mathrm{M}_{2\times 2}(\mathbb{R})$. What are the dimensions of $W_1 + W_2$ and $W_1 \cap W_2$? *Justify your answers.*

2. Let $\mathrm{P}_3(\mathbb{R})$ be the real vector space consisting of all polynomials of degree at most 3 with real coefficients, and let $T\colon \mathrm{P}_3(\mathbb{R}) \to \mathrm{P}_3(\mathbb{R})$ be the map defined by

$$T(f) = 2f - 3f'$$

for $f \in \mathrm{P}_3(\mathbb{R})$. Here f' denotes the derivative of f.

(a) (8 points) Prove that T is a linear transformation.

(b) (12 points) Let $\beta = \{1, x+1, x^2, x^3 - 1\}$, which is an ordered basis for $\mathrm{P}_3(\mathbb{R})$. Find the matrix representation $[T]_\beta$ of T in β.

3. Let $\{u, v, w\}$ be a basis for $\mathbb{R}^3$. Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $T(u) = 5u - 3w$, $T(v) = 7u + 4v + 14w$, and $T(w) = u + w$.

(a) (10 points) Find all eigenvalues of T.

(b) (10 points) Is T diagonalizable? *Justify your answer.*

4. Let n be a positive integer, and let V be an n-dimensional complex inner product space. For all $v, w \in V$, the inner product of v and w is denoted by $\langle v, w \rangle$.

(a) (10 points) Suppose that $\{u_1, \ldots, u_n\}$ is an orthonormal basis for V. Prove that for every $v \in V$, $v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i$.

(b) (10 points) Let T be a self-adjoint operator on V. Prove that every eigenvalue of T is a real number.

5. (20 points) Let A be the real matrix

$$\begin{bmatrix} 3 & -1 & 3 \\ 1 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find an invertible 3×3 real matrix P such that $P^{-1}AP$ is in Jordan canonical form.