臺灣綜合大學系統 113 學年度學士班轉學生聯合招生考試試題

科目名稱	線性代數	類組代碼	A07/C11
		科目碼	A0702

※本項考試依簡章規定所有考科均「不可」使用計算機。 本科試題共計 2 頁

Notations:

(1) $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^n$ is the set of all n-tuples with entries from $\mathbb{R}$.

(2) $P_n(\mathbb{R})$ is the set of all polynomials with degree less than or equal to n and coefficients in $\mathbb{R}$.

(3) $\beta_n := \{1, x, x^2, \cdots, x^n\}$ is the standard ordered basis of $P_n(\mathbb{R})$.

(4) $M_{m\times n}(\mathbb{R})$ is the set of all $m \times n$ matrices with entries from $\mathbb{R}$.

Problems:

1. (10%) Prove or disprove the following statement:
Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation. Then T is one-to-one if and only if the null space (or called kernel) of T is spanned by $(0, 0, 0) \in \mathbb{R}^3$.

2. (10%) Define $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by $T(f)(x) = 2f'(x) + \int_0^x f(t)dt$. Find the matrix representation of T in the stantard ordered bases β_2 and β_3.

3. Let

$$W_1 = \left\{ \begin{bmatrix} a & c \\ c & b \end{bmatrix} \middle| a, b, c \in \mathbb{R}. \right\}$$

and

$$W_2 = \left\{ \begin{bmatrix} a & 0 \\ b & -a \end{bmatrix} \middle| a, b \in \mathbb{R}. \right\}.$$

(a) (7%) Find the dimension of $W_1 \cap W_2$. Justify your answer.

(b) (8%) Find the dimension of $W_1 + W_2$. Justify your answer.

4. Let $A \in M_{m\times n}(\mathbb{R})$ and $P \in M_{n\times n}(\mathbb{R})$.

(a) (7%) Show that rank$(AP) \leq$ rank(A).

(b) (8%) Show that rank(AP)=rank(A) if P is invertible.

5. (10%) Prove or disprove the following statement: If $A \in M_{2\times 2}(\mathbb{R})$ satisfies

$$A^3 = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix},$$

then A is invertible.

6. (a) (10%) Let $A \in M_{3\times 3}(\mathbb{R})$. Suppose that A has three eigenvalues 1, 2 and 3. Show that A is diagonalizable.

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(b) (10%) Let $A \in M_{5\times 5}(\mathbb{R})$. Suppose that

$$\det(A - \lambda I) = -(\lambda - 4)^2(\lambda - 5)^3.$$

Either prove that A is diagonalizable or find all possible Jordan canonical forms of A.

7. Let $C[-1,1]$ be the space with all continuous functions on $[-1,1]$. Define an inner product on $C[-1,1]$ by

$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)dx \text{ for } f, g \in C[-1,1]$$

and set

$$||f|| := \sqrt{\langle f, f\rangle}.$$

(a) (10%) Find an orthonormal basis of $P_2(\mathbb{R})$ by using the Gram-Schmidt process applied to β_2.

(b) (10%) Let $h(x) = x^3$. Find a polynomial $u \in P_2(\mathbb{R})$ such that

$$||h - f|| \geq ||h - u|| \text{ for all } f \in P_2(\mathbb{R}).$$